# The Representation Theory of the Symmetric Groups: An Inductive Approach

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## 1. Introduction

As the title suggests, in this paper, we outline an inductive approach to the representation theory of the symmetric groups, based on the paper "A New Approach to the Representation Theory of the Symmetric Groups" written by Vershik and Okounkov [VO05]. Instead of simply regurgitating the work of Vershik and Okounkov, the idea behind this paper is to use a complicated-enough example (namely, we will be looking at the standard representation of $S_4$) to illustrate the results presented in this paper. Having this companion example to work through will (hopefully) make everything more easily understood and digestible.

The traditional approach used to study the representation theory of $S_n$ ([FH91]) is unwieldy for multiple reasons: it involves induced representations, which are difficult to decompose into irreducibles; the correspondence between diagrams and representations is rather unnatural; this approach obscures some very important properties of $S_n$. In particular, our approach makes use of the natural chain of embeddings $S_{n-1} \hookrightarrow S_n$ (so that the representation theory of $S_n$ relies on that of $S_{n-1}$) and the fact that symmetric groups are Coxeter groups. Lastly, our inductive approach is preferable because it can be generalized to other such inductive chains of groups.

## 2. Representations of $S_4$

Throughout we assume a basic knowledge of representation theory—this section is devoted to recalling the representation theory of $S_n$ for $n \leq 4$. Namely, we will just list out the relevant character tables and name the various irreducibles. For $S_1$, we just have the trivial representation, $T$, and for $S_2$, we let $U_1$ be the trivial representation and let $U_2$ be the alternating representation:

| $S_1$ | (1) |
|---|---|
| $T$ | 1 |

| $S_2$ | (1) | (12) |
|---|---|---|
| $U_1$ | 1 | 1 |
| $U_2$ | 1 | $-1$ |

We let $V_1, V_2, V_3$ denote the trivial, standard, and alternating representations of $S_3$:

| $S_3$ | (1) | (12) | (123) |
|---|---|---|---|
| $V_1$ | 1 | 1 | 1 |
| $V_2$ | 2 | 0 | $-1$ |
| $V_3$ | 1 | $-1$ | 1 |

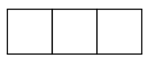
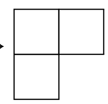
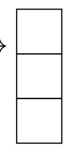
| $S_4$ | (1) | (12) | (123) | (1234) | (12)(34) |
|---|---|---|---|---|---|
| $W_1$ | 1 | 1 | 1 | 1 | 1 |
| $W_2$ | 3 | 1 | 0 | $-1$ | $-1$ |
| $W_3$ | 2 | 0 | $-1$ | 0 | 2 |
| $W_4$ | 3 | $-1$ | 0 | 1 | $-1$ |
| $W_5$ | 1 | $-1$ | 1 | $-1$ | 1 |

The character table for $S_4$ is given above, where $W_1$ and $W_5$ are respectively the trivial and alternating representations of $S_4$, $W_2$ is the standard representation, and $W_4 \cong W_2 \otimes W_5$.

## 3. The Main Theorem

**Theorem 3.1.** *The irreducible representations of $S_n$ are in bijective correspondence with Young diagrams with $n$ boxes. Moreover, each irreducible representation of $S_n$ has a basis whose elements are indexed by standard Young tableaux of the same shape with $n$ boxes.*

**Example 3.2.** In the case of $S_3$, we will see that

$V_1 \longleftrightarrow$  $V_2 \longleftrightarrow$  $V_3 \longleftrightarrow$ 

## 4. The Branching Graph

Consider an inductive chain of finite groups

$$\{1\} = G(0) \subset G(1) \subset G(2) \subset \dots$$

For each $n$, let $G(n)^\wedge$ denote the set of isomorphism classes of irreducible complex representations of $G(n)$. If $\lambda \in G(n)^\wedge$, let $V^\lambda$ be the corresponding $G(n)$-representation.

**Definition 4.1.** The *branching graph* of

$$\{1\} = G(0) \subset G(1) \subset G(2) \subset \dots$$

is the multigraph with

(1) Vertex set $\coprod_{n \geq 0} G(n)^\wedge$;

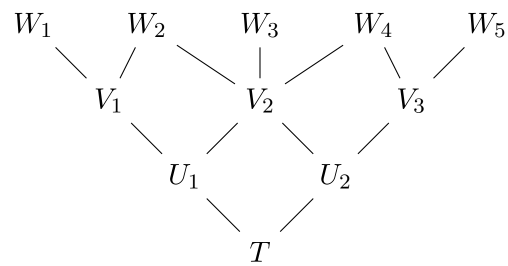
(2) Edges given in the following way: Let $\lambda \in G(n)^\wedge$ and let $\mu \in G(n-1)^\wedge$. Consider $V^\lambda$ as a $G(n-1)$-module, and decompose it into irreducibles. Let $k$ be the multiplicity of $\mu$ in $\lambda$ as a $G(n-1)$-module (i.e. the number of times $\mu$ appears in this direct sum decomposition). Then, $\mu$ and $\lambda$ are connected by $k$ directed edges with source $\mu$ and target $\lambda$. If $k$ is nonzero, we write $\mu \nearrow \lambda$. If $i < n-1$, then no vertex in the $i$th level of the graph is connected to any vertex in $G(n)^\wedge$.

For any representations $\mu$ and $\lambda$ in the branching graph, we say $\mu \subset \lambda$ if there exists a path from $\mu$ to $\lambda$. If all multiplicities are 0 or 1 (so we have a graph, rather than a multigraph), then we say the branching is *simple.*

**Theorem 4.2.** *The branching of $S_n$ is simple.*

We omit the proof and instead illustrate the above definition and theorem with $S_4$. By examining the character tables of $S_4$, it is not difficult to determine its branching: starting with $S_n$ one can literally omit the columns of the character table containing elements not in $S_{n-1}$; this will give the characters of representations in the $n$th level of the branching graph as $S_{n-1}$-modules. From there, we know how to decompose everything into a direct sum of irreducibles of $S_{n-1}$; the branching follows inductively.

**Example 4.3.** The branching of $S_4$ looks like:




**Definition 4.4.** For chains of groups with simple branching, the decomposition of a $G(n)$-module into $G(n-1)$-irreducibles is canonical. Then, inductively, for each $\lambda \in G(n)^\wedge$, we get a canonical decomposition of $V^\lambda$ into $G(0)$-modules:

$$V^\lambda = \bigoplus_T V_T,$$

where the sum is indexed by all chains (i.e. increasing paths from the trivial $G(0)$-module to $\lambda$ in the branching graph) $T = \lambda_0 \nearrow \lambda_1 \nearrow \ldots \nearrow \lambda_n$, where $\lambda_i \in G(i)^\wedge$ and where $\lambda_n = \lambda$. Up to scaling, this process gives us a basis $\{v_T\}$ for $V^\lambda$ indexed by all paths from the trivial $G(0)$-module to $\lambda$; this is called the *Gelfand-Tsetlin* basis. The disjoint union of all such Gelfand-Tsetlin bases is called the *Young basis*, which we denote by $\mathcal{Y}$.

**Example 4.5.** We compute the Gelfand-Tsetlin basis of the standard representation of $S_4$, $W_2$. Recall that $W_2 = \{(a_1, a_2, a_3, a_4) \mid \sum a_i = 0\}$. Restricting our $S_4$-action to $S_3$, it is easy to check that

$$W_2 = \text{span}\{(1,1,1,-3)\} \oplus \text{span}\{(1,-1,0,0),(0,1,-1,0)\},$$

where the first $S_3$-module is isomorphic to the trivial representation; the second to the standard representation of $S_3$. Restricting again to $S_2$, the above decompose into span$\{(1,1,1,-3)\}$ and

$$\text{span}\{(1,-1,0,0),(0,1,-1,0)\} = \text{span}\{(1,-1,0,0)\} \oplus \text{span}\{(1,1,-2,0)\},$$

where the first module is the alternating representation of $S_2$; the other is the trivial representation. This implies that $W_2$ decomposes canonically into the following direct sum of $S_1$-modules:

$$W_2 = (1,1,1-3) \oplus (1,1,-2,0) \oplus (1,-1,0,0).$$

Thus, $\{(1,1,1-3),(1,1,-2,0),(1,-1,0,0)\}$ is our Gelfand-Tsetlin basis for $W_2$. Moreover, we note that each vector corresponds to a path in the branching graph:

(a) $(1,1,1-3)$ corresponds to $T \nearrow U_1 \nearrow V_1 \nearrow W_2$

(b) $(1,1,-2,0)$ corresponds to $T \nearrow U_1 \nearrow V_2 \nearrow W_2$

(c) $(1,-1,0,0)$ corresponds to $T \nearrow U_2 \nearrow V_2 \nearrow W_2$.

## 5. The Gelfand-Tsetlin Algebra and Young-Jucys-Murphys Elements

**Definition 5.1.** Denoting by $Z(n)$ the center of $\mathbb{C}G(n)$, the $n$th *Gelfand-Tsetlin (sub)algebra* of the inductive family of groups is the algebra generated by

$$Z(1), Z(2), \ldots, Z(n).$$

This is denoted $GZ(n)$. It can be shown that $GZ(n)$ is a maximal commutative subalgebra of $\mathbb{C}G(n)$ and that it is the algebra of all operators that are diagonal in the GZ-basis.

From here on out, we let $G(n) = S_n$ and restrict to this case. We look at a special basis of $GZ(n)$:

**Definition 5.2.** The *Young-Jucys-Murphys (YJM) elements* $X_1, \ldots, X_n$ are the elements

$$X_i := (1i) + \ldots + (i-1\ i)$$

(with $X_1 = 0$) of $\mathbb{C}S_n$.

Because $GZ(n)$ is the algebra of all operators that are diagonal in the GZ-basis, the Young basis is a common eigenbasis of the YJM elements. Thus, the following definition makes sense:

**Definition 5.3.** For any $v$ in the Young basis, the *weight* of $v$ is the $n$-tuple

$$\alpha(v) := (a_1, \ldots, a_n) \in \mathbb{C}^n,$$

where $a_i$ is the eigenvalue of $v$ under $X_i$. The *spectrum* of $S_n$ is the set of all tuples:

$$\mathrm{Spec}(n) = \{\alpha(v) \mid v \in \mathcal{Y}_n\}.$$

The weight of a vector determines it up to scaling. By definition, we see that $\mathrm{Spec}(n)$ is in natural bijection with the set of all paths in the branching graph of $S_n$ (recall that each vector in $\mathcal{Y}$ is indexed by a path in the branching graph). Let $\sim$ be the equivalence relation on $\mathrm{Spec}(n)$ equating two weights if their corresponding paths have the same target in the $n$th level of the branching graph. In other words, for any $\alpha, \beta \in \mathrm{Spec}(n)$, $\alpha \sim \beta$ if $v_\alpha$ and $v_\beta$ belong to the same irreducible representation of $S_n$.

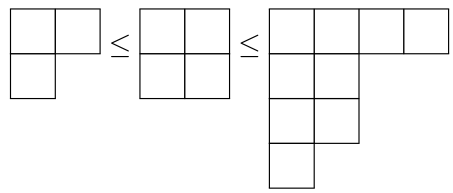
**Example 5.4.** We calculate the spectrum of the standard representation of $S_4$ with respect to the YJM elements. Our YJM elements are $X_1 = 0$, $X_2 = (12)$, $X_3 = (13)+(23)$ and $X_4 = (14)+(24)+(34)$. For $v_1 = (1,1,1-3)$, it is just a computation to see that

$$X_1 v_1 = 0,\ X_2 v_1 = v_1,\ X_3 v_1 = 2v_1,\ X_4 v_1 = -v_1,$$

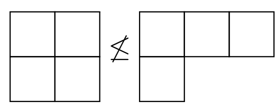
so $\alpha(v_1) = (0,1,2,-1)$. Similarly, we may compute that for $v_2 := (1,1,-2,0)$, $\alpha(v_2) = (0,1,-1,2)$; for $v_3 = (1,-1,0,0)$, $\alpha(v_3) = (0,-1,1,2)$. Under the equivalence relation $\sim$, we see that $\alpha(v_i) \sim \alpha(v_j)$ for all $i, j$, since the $v_i$'s all belong to the same irreducible representation of $S_4$.

## 6. Young Diagrams, Tableaux, and Young's Lattice

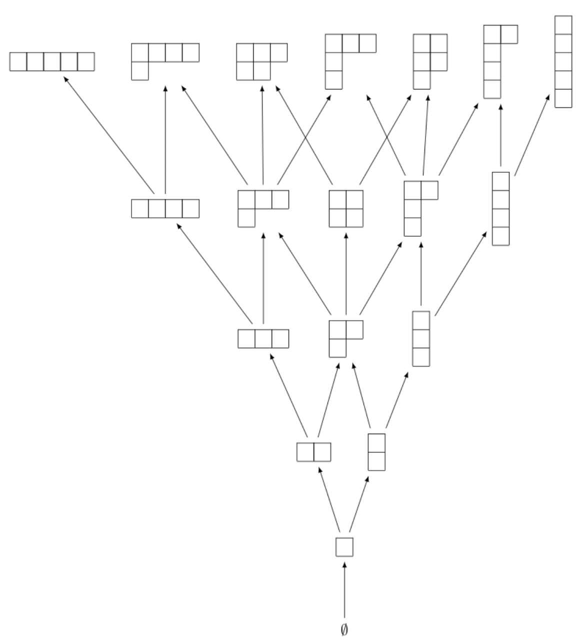
Begin by noting that there is a partial order on the set of all Young diagrams given by inclusion. In particular, given two partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ (with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$) and $\mu = (\mu_1, \ldots, \mu_j)$, $\lambda \leq \mu$ if $\lambda_1 \leq \mu_{r_1}$ for some $r_1$, $\lambda_2 \leq \mu_{r_2}$ for $r_2 > r_1$, and so on. This is perhaps easier to represent pictorially:



but

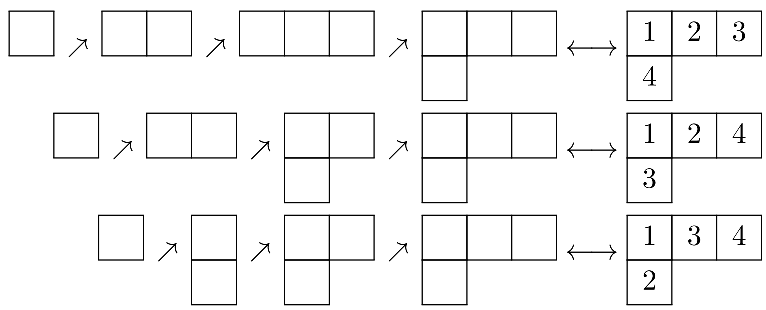


This partial order gives us a lattice, called *Young's lattice*, which we denote by $\mathbb{Y}$. The nodes in this lattice are given by all possible Young diagrams, and we draw and edge between two nodes if they differ by exactly one box and if one is less than the other under our order. We note that given two diagrams $\lambda$ and $\mu$, $\lambda \leq \mu$ if and only if there exists a path from $\lambda$ to $\mu$ in Young's lattice.




Now, we describe a bijection between standard Young tableaux and paths in Young's lattice. The bijection is as follows. Given some path in the lattice starting at the bottommost vertex, note that in each step of the path, we add a box to the diagram. For the $i$th step in the path, place the number $i$ in the box we add.

**Example 6.1.** We illustrate this for tableaux of shape $(3, 1)$. We have the following correspondences:




## 7. Content Vectors

**Definition 7.1.** A vector $(a_1, \ldots, a_n) \in \mathbb{C}^n$ is called a *content vector* if

(1) $a_1 = 0$;

(2) for $m > 1$, if $a_m > 0$, then $a_i = a_m - 1$ for $i < m$; if $a_m < 0$ then $a_i = a_m + 1$ for $i < m$;

(3) if $a_i = a_j = a$ for $a_i < a_j$, then $\{a - 1, a + 1\} \subset \{a_{i+1}, \ldots, a_{j-1}\}$.

The set of all content vectors of length $n$ is denoted $\mathrm{Cont}(n)$. There is an equivalence relation $\approx$ on $\mathrm{Cont}(n)$ defined by $\alpha \approx \beta$ if $\alpha$ is a permutation of $\beta$, where $\alpha, \beta \in \mathrm{Cont}(n)$ [Kim18].

**Definition 7.2.** Given a Young diagram, we may assign to each box in the diagram a number, called the *content* of the box. To do so, we assign coordinates to the Young diagram: boxes in the $i$th column have $x$-coordinate $i-1$, and boxes in the $j$th row have $y$-coordinate $j-1$. Then, the content of some box is the $x$-coordinate of the box minus the $y$-coordinate of the box.

**Example 7.3.** The contents of the boxes in $(2, 2)$ are given below:

| | |
|---|---|
| 0 | 1 |
| $-1$ | 0 |

**Theorem 7.4.** *To each Young tableaux with $n$ boxes, we may associate a content vector of length $n$. To do so, let box $i$ be the box of the tableaux with the number $i$ in it. Then, the vector in question is simply the vector in $\mathbb{Z}^n$ whose $i$th coordinate is the content of the $i$th box. This is a content vector. Moreover, this correspondence is a bijection, and $\alpha \approx \beta$ if and only if the corresponding paths in Young's lattice end at the same diagram (i.e. the corresponding tableaux have the same shape).*

The proof is omitted, and instead we calculate the content vectors of the tableaux from Example 6.1.

**Example 7.5.** The contents of (3,1) are given by

$$\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline -1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

Hence, the associated content vectors are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \longleftrightarrow (0, 1, 2, -1)$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \longleftrightarrow (0, 1, -1, 2)$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \longleftrightarrow (0, -1, 1, 2)$$

Now, a little more work (specifically one needs to study Coxeter generators) and some induction gives the following theorem:

**Theorem 7.6.** *Spec*$(n) \subset$ *Cont*$(n)$

The following result is also important:

**Proposition 7.7.** *If* $\alpha \in$ *Spec*$(n)$ *with* $\alpha \approx \beta$ *for* $\beta \in$ *Cont*$(n)$*, then* $\beta \in$ *Spec*$(n)$ *and* $\alpha \sim \beta$*.*

## 8. The Main Theorem (reprise)

The payoff of all of the above work is the following theorem, which is the main result from earlier.

**Theorem 8.1.** *Young's lattice* $\mathbb{Y}$ *is the branching graph of the symmetric groups, and the spectrum of the Gelfand-Tsetlin algebra* $GZ(n)$ *corresponds bijectively to the set of paths in* $\mathbb{Y}_n$*, which is just the set of standard Young tableaux with* $n$ *boxes. More specifically,* *Spec*$(n) =$ *Cont*$(n)$*, and their respective equivalence relations coincide:* $\sim = \approx$*.*

*Proof.* By Theorem 7.4, $\mathrm{Cont}(n)/\approx$ is the set of classes of tableaux that have the same shape. Therefore,

$$\#(\mathrm{Cont}(n)/\approx) = p(n).$$

Now, Proposition 7.7 tells us that each equivalence class in $\mathrm{Cont}(n)/\approx$ either contains a subset of $\mathrm{Spec}(n)$ or has empty intersection with $\mathrm{Spec}(n)$. Moreover, if the class in $\mathrm{Cont}(n)/\approx$ meets $\mathrm{Spec}(n)$, then it is a subset of some class in $\mathrm{Spec}(n)/\sim$. Now, recall that the number of irreducible representations of $S_n$, $\#S_n^\wedge$, is given by $\#(\mathrm{Spec}(n)/\sim)$. Since the number of irreducible representations of a group is equal to the number of conjugacy classes, and since conjugacy classes in $S_n$ are determined by cycle type, it follows that

$$\#(\mathrm{Spec}(n)/\sim) = \#S_n^\wedge = p(n),$$

as number of possible cycle types in $S_n$ is equal to $p(n)$. Ergo, each class of $\mathrm{Cont}(n)/\approx$ coincides with one of the classes in $\mathrm{Spec}(n)/\sim$, implying

$$\mathrm{Spec}(n) = \mathrm{Cont}(n) \qquad \text{and} \qquad \sim=\approx .$$

This is exactly what we wanted to show. □

We return to our example of the standard representation $W_2$ of $S_4$ for a final time:

**Example 8.2.** We calculated in Example 5.4 the spectrum of the Young basis of $W_2$ with respect to the YJM elements. Specifically, we saw that

$$\begin{aligned} (1,1,1-3) &\longleftrightarrow (0,1,2,-1) \\ (1,1,-2,0) &\longleftrightarrow (0,1,-1,2) \\ (1,-1,0,0) &\longleftrightarrow (0,-1,1,2). \end{aligned}$$

These are elements of Spec(4), and they all belong to the same equivalence class under $\sim$. By the previous results, we know that each of these is a content vector associated to some tableaux with 4 boxes. Moreover, we should see that each of the associated tableaux have the same shape, since $\sim=\approx$ and since two tableaux have the same shape if and only if their associated content vectors are equal under $\approx$. We saw earlier in Example 7.5 that

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \longleftrightarrow (0,1,2,-1)$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \longleftrightarrow (0,1,-1,2)$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \longleftrightarrow (0,-1,1,2).$$

Therefore,

$$(1,1,1-3) \longleftrightarrow (0,1,2,-1) \longleftrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$$

$$(1,1,-2,0) \longleftrightarrow (0,1,-1,2) \longleftrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$(1,-1,0,0) \longleftrightarrow (0,-1,1,2) \longleftrightarrow \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$$

gives us the correspondence between the Young basis of $W_2$ and Young tableaux. Moreover, because $\sim=\approx$, this tells us that the representation $W_2$ of $S_4$ corresponds to the Young diagram given by (3,1),

$$W_2 = \mathrm{span}\{(a_1,a_2,a_3,a_4) \in \mathbb{C}^4 \mid \textstyle\sum a_i = 0\} \longleftrightarrow \begin{array}{|c|c|c|}\hline \phantom{0} & \phantom{0} & \phantom{0} \\ \hline \phantom{0} \\ \cline{1-1}\end{array}$$

## References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