Isomorphism: the Mathematician's Metaphor

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Introducing Isomorphism

Asked if he believes in one God, a mathematician answered: "Yes, up to isomorphism."
–Godfrey H. Hardy

What does it mean for two things to be *equal*? And, more practically, how do we discern when two objects are equal and when they are not? According to Leibniz, two equal objects share all the same properties (this is known as *Leibniz's Law* or as the *Indiscernibility of Identicals*) [6]. From this point of view, objects are, at least in some way, determined by their properties. This notion, regardless of whether or not it is "correct," is not entirely alien. In the physical world, we differentiate objects on a daily basis by observing and measuring—with sight, touch, smell, scale, etc.—their various physical properties. In mathematics, this Wittgensteinian notion is made formal via the *Yoneda Lemma*, which states that an object is entirely determined by the ways in which it relates to itself and all other objects. Thus, in order to discern one distinct object from another, it suffices to find some property that one object exhibits and the other does not. But given some specific property, how are such comparisons even made? It often helps to "forget" about every property of the object other than the distinguished one and view two objects as "equal" if they both share this distinguished property. From this point of view, many objects that are distinct at face value turn out to be indistinguishable. This blurring of equality is at the heart of what mathematicians call *isomorphism*.

The notion of isomorphism is ubiquitous in the study of mathematics, and its precise definition can vary depending on the context. The word itself comes from Ancient Greek: *isos*, meaning equal, and *morphe*, meaning form. Fundamentally, isomorphism can be thought of as the mathematician's metaphor—a likening of some aspect of one object to an analogous aspect of an other. Mathematics is often perceived to be a literal, objective science, and mathematicians are stereotyped as exacting pedants obsessed with numbers and formulas. But in many ways, the study of mathematics is much like the exploration of poetry.

Whether in mathematics or poetry, the concept of equality is rooted in the art of comparison. While poetry is not necessarily concerned with configuring equalities for their own sake, poets use literary devices such as analogy and metaphor to compare or relate, figuratively, objects or ideas that are literally inapposite. Analogy is more objective and literal and more closely hews to comparison—likening things and ideas that are more obviously similar. Metaphor, which is more subjective and figurative, derives its power from choosing what and how to compare and contrast. Thus, isomorphism is most appropriately analogized to metaphor because these two concepts involve the challenge of discerning similarity amidst difference. In poetry, metaphor is not simply an exercise to determine if disparate objects or ideas are equivalent but rather is a literary strategy aimed at suggesting an idea or feeling that is otherwise difficult to capture or express.

For instance, in the following excerpt from *The Love Song of J. Alfred Prufrock*, T. S. Eliot paints a dingy, urban scene by likening the smog that settles on a house to a cat [2]:

The yellow fog that rubs its back upon the window-panes,
The yellow smoke that rubs its muzzle on the window-panes,
Licked its tongue into the corners of the evening,
Lingered upon the pools that stand in drains,
Let fall upon its back the soot that falls from chimneys,

Slipped by the terrace, made a sudden leap,
And seeing that it was a soft October night,
Curled once about the house, and fell asleep.
–T. S. Eliot

Rather than just describing the polluted fog in literal detail, through this metaphor, Eliot conjures an image of the fog that comes to life by capturing its very movement. The fog and the cat are not equal things, but in this stanza, the properties distinguishing them are "forgotten" and the difference between them smudged because their one related property—the similarity in their movement—is the focus. Referring to the fog as cat-like captures and visually portrays its lithe movement as it curls and presses against the window panes. In this way, the fog becomes animate, and it becomes almost difficult to imagine fog as not feline. Similarly, a cat's ability to insinuate itself into any space and settle gracefully into any crevice is uncannily fog-like.

Thus, the beauty and bounty of metaphor is that in associating two otherwise unrelated ideas or objects, the comparison offers a deeper intrinsic understanding of each. At the same time, the comparison of disparate objects or ideas allows for a more expansive rendering of both, as one's intuitions and associations about one object inform how one views the other via the metaphor. Through Eliot's deft yoking, the metaphor itself, once surprising given how unrelated a cat and smog are to each other, now seems obvious, inevitable even. The very meaning and feeling of the stanza—the sooty dinginess; the feline solitariness; and yet the gentle coziness of a fall night graced by the conjuring of the idea of a slumbering cat—is ultimately entangled with the comparison.

The notion of isomorphism in mathematics similarly allows us to compare and draw connections between seemingly unrelated objects. These mathematical metaphors often have significant practical value in that they allow us to solve problems that were previously either incredibly difficult to solve or otherwise intractable. By showing an unknown object is isomorphic to something that is well-understood, we reveal that the first object shares the relevant, well-known properties of the second one. Beyond being a useful tool in solving problems, an isomorphism can be beautiful in and of itself, just as the metaphor in *Prufrock* is appealing independent of the context of the larger poem.

If the idea of isomorphism is, at its core, the equation of objects sharing some specific property, then it is natural to ask how one should go about deciding which property to distinguish. Fundamentally, the more elementary question is: how does one go about creating definitions in mathematics? After all, a definition is simply a label we create for the objects that share some specified collection of properties. Analogously, a poet might ask: what is it about what I am describing that I should metaphorize?

Creating "the right" definitions extremely difficult and can take years of research to do so in a particular case, let alone in general. Often, it is the case that we notice peculiar similarities between disparate objects, and our goal might be to understand what exactly it is about these objects that is the same. In the next section, we will see that the permutations of the numbers 1, 2, and 3 are "the same" as the symmetries of an equilateral triangle. This is a beautiful (and, if not seen before, surprising) fact, since, at first, permutations seemingly have little to do with triangles. Perhaps even more beautiful is the kernel of this connection—what it is, fundamentally, about the permutations and symmetries that makes them similar. By way of *group theory*, the aforementioned equivalence can be reduced to a singular idea: *symmetry*.

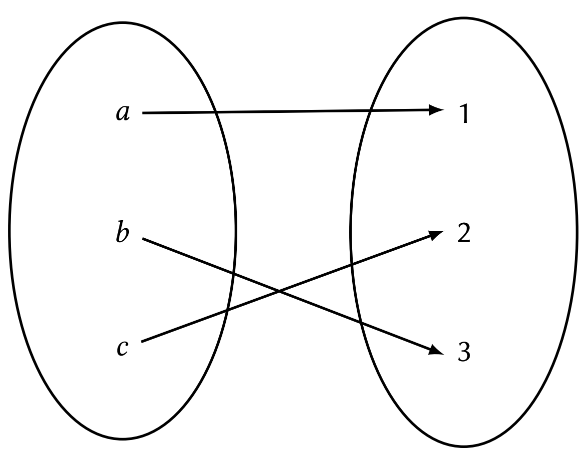
Thus, deciding what it should mean for two objects to be isomorphic is fundamentally the same challenge as deciding how to go about creating some sort of universal definition that encompasses all of the objects we perceive to be similar in some way. Doing so requires distilling these perceived similarities into a fundamental singularity. One such example of this sort of unification of theories is considered to be the most triumphant achievements in modern mathematics: Grothendieck's definition of a scheme. The notion of a scheme, which we will introduce briefly and informally in one of the later sections of this paper, is a synthesization of connections between geometry, algebra, and number theory into a singular object that encompasses and connects all three theories. In this way, isomorphism can transcend metaphor in that it can move past a simple comparison and ultimately be used to describe what something really is and even create new definitions and concepts.

Symmetry: an Example

Tyger Tyger, burning bright,
In the forests of the night;
What immortal hand or eye,
Could frame thy fearful symmetry?
–*William Blake,* The Tyger [1]

Before we get ahead of ourselves discussing symmetry and groups, perhaps it is best to begin our discussion with a simple example of isomorphism. Consider the following two sets: $\{a, b, c\}$ and $\{2, 3, 5\}$. From a literal perspective, these sets are entirely different. One set consists of letters, while the other is made up of numbers. But, by simply counting their elements, we immediately see a connection between the two sets: they are equinumerous, both consisting of three elements. Thus, if we were to somehow forget the names of the constituent elements of each set, the two sets would look identical. This link between $\{a, b, c\}$ and $\{1, 2, 3\}$ is our first example of an isomorphism: while the sets themselves are not literally equal, they are figuratively so—they have the "same form" in that they have the same number of elements.

Formally, an isomorphism between two sets A and B is a mapping $f : A \to B$ that is *injective*—no two elements of A are mapped to the same element of B—and *surjective*—every element of B can be written as $f(a)$ for some a in A. Maps that are both injective and surjective are called *bijective*. From this point of view, the bijection given by



is an isomorphism between $\{a, b, c\}$ and $\{1, 2, 3\}$ (note that there are $3! = 6$ total possible isomorphisms $\{a, b, c\} \longrightarrow \{1, 2, 3\}$; the one described above is just one of them). If we consider only finite sets (only to avoid having to discuss sets containing infinitely many elements), then we see that, from this point of view, two sets are isomorphic if and only if they have the same number of elements.

We now discuss the group theoretic isomorphism mentioned in the introduction. Consider the set of permutations of the numbers 1, 2, and 3, i.e., the bijective maps $\{1, 2, 3\} \longrightarrow \{1, 2, 3\}$. Denote this set by S_3, and note that it has 3! elements. We may represent each permutation as a 2×3 matrix of numbers, where a permutation $\sigma : \{1, 2, 3\} \longrightarrow \{1, 2, 3\}$ is represented by the matrix

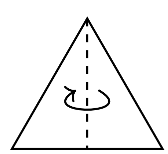
$$\begin{bmatrix} 1 & 2 & 3 \\ \sigma(1) & \sigma(2) & \sigma(3) \end{bmatrix}.$$

For example, the permutation taking 1 to 2, 2 to 3, and 3 to 1 is represented by

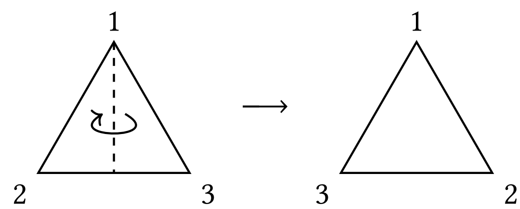
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

As a set, we saw in the above that S_3 is isomorphic to any other set with 6 elements. Upon further examination, however, we see that, because they are functions, any two elements σ and τ of S_3 can be composed to give a new element $\sigma \circ \tau$ of S_3. This law of composition is important because it allows us to combine and relate the elements of S_3 to each other. Upon closer examination, we see that this operation also has a few interesting properties: the composition is associative; there exists an identity permutation (i.e., a permutation id such that $\sigma \circ \mathrm{id} = \sigma = \mathrm{id} \circ \sigma$ for any σ in S_3); and every element of S_3 has an inverse (i.e., for each σ in S_3 there exists a permutation τ such that $\sigma \circ \tau = \mathrm{id} = \tau \circ \sigma$).

Now, let D_3 denote the set of symmetries of an equilateral triangle, i.e., the set of all isometries (bijective, distance-preserving functions) of the plane mapping the triangle to itself. There are 6 total symmetries. Three of the symmetries are rotations about the center of the triangle by 0°, 120°, and 240°; the other three are reflections about the lines connecting a vertex to the center of the triangle (one of the reflections is pictured below). Since D_3 is a set of functions, it likewise comes equipped with a law of composition; this operation also has the same notable properties as the operation on S_3.



While D_3 and S_3 seem to be fundamentally different, with the former being fundamentally geometric and the latter combinatorial, it turns out that, in some sense, they are the same. One easy way to see this is as follows. Label the vertices of the triangle 1, 2, and 3. Then, each symmetry f in D_3 corresponds to a permutation σ_f in S_3 given by the way in which f permutes the vertices of the triangle. In can be checked (albeit somewhat painstakingly) that this correspondence is bijective and that for all f and g in D_3 we have $\sigma_{f\circ g} = \sigma_f \circ \sigma_g$, and vice versa. For example, label the vertices of the triangle as in the figure below.



We see that the symmetry illustrated above corresponds to the permutation

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix},$$

since the vertex 1 is fixed and the vertices 2 and 3 are interchanged. More abstractly, we see that S_3 and D_3 are the same up to relabeling of their elements. Not only are S_3 and D_3 isomorphic as sets (they both have 6 elements), but their composition laws are the same. If we were to forget what the elements of S_3 and D_3 are and instead view them as abstract sets equipped with laws of composition (with the aforementioned properties), then S_3 and D_3 would "look" identical.

This hints at a deeper, more intrinsic connection between S_3 and D_3. Rather than simply being "the same as sets," it is the accompanying composition law that unites S_3 and D_3. At its heart, this similarity is borne from the notion of *symmetry*. A *symmetry* of an object should be thought of as an action—a map that preserves the object—rather than a property an object has. Formally, it is a "mapping of the object onto itself which preserves [its] structure" [10]. Two symmetries can be composed; this composition has exactly the same properties as composition laws on S_3 and D_3. From this point of view, the symmetries of a set are simply the bijections from the set to itself. In other words, S_3 is simply the set of symmetries of the set $\{1, 2, 3\}$. This deeper connection between S_3 and D_3 leads to the definition of one of the most fundamental objects in mathematics: the *group*. Formally, a group is a set equipped with a law of composition satisfying the same properties that we have been discussing in this section. Informally, a group can be thought of as the set of symmetries of some object (there are ways of making this precise). In this way, the definition of the group—this "immortal hand"—transcends the humble comparisons which led to its creation and truly captures the essence of what S_3 and D_3 *are*.

Categorical Imperatives

Act only according to that maxim whereby you can at the same time will that it should become a universal law.
–*Immanuel Kant,* Groundwork for the Metaphysics of Morals [5]

Having seen some elementary examples of isomorphisms, we now return to a much more general point of view to introduce some basic *category theory*. The language of category theory will not only serve us well in the sequel, but it will also help to reinforce some of the ideas introduced in the previous section. Category theory makes precise the perceived relationships between entire collections of objects. The following is adapted from [7] and [8].

A *category* is defined to be a collection of *objects* (usually denoted $X, Y, \ldots$) which are related to one another by *morphisms* (usually denoted using labeled arrows $f : X \to Y$). The collection of morphisms between two objects X and Y is denoted by $\mathrm{Mor}(X, Y)$. Morphisms can be *composed*: given objects X, Y, and Z and morphisms $f : X \to Y$ and $g : Y \to Z$, there is a morphism $X \to Z$

called $g \circ f$. This composition law is *associative*, so for morphisms $f : W \longrightarrow X$, $g : X \longrightarrow Y$, and $h : Y \longrightarrow Z$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

In other words, there is no ambiguity in writing $h \circ g \circ f$. Finally, for each object X in the category, there is an *identity morphism* $\mathrm{id}_X : X \longrightarrow X$ such that for any morphism $f : X \longrightarrow Y$ we have $f = f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f$. We say two objects X and Y of a category are *isomorphic* if there exist morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. The *isomorphism class* of an object X is simply the collection of all objects in the category isomorphic to X.

One of the most basic examples of a category is the category of sets, denoted Set. Its objects are, as its name suggests, sets, and the morphisms between sets are the honest-to-god functions $f : X \longrightarrow Y$ associating to each element of x a corresponding element of Y, which we denote by $f(x)$. From this point of view, it is straightforward to check that the category-theoretic notion of isomorphism coincides with the notion of isomorphism of sets introduced in the previous section: two sets are isomorphic if and only if they have the same cardinality.

Another example of a category is the one whose objects are topological spaces and whose morphisms are continuous mappings between them. Roughly, a *topology* on a space is a way of measuring how close two points of the space are. Here, two spaces are isomorphic if and only if there is a bijection between them and the topologies on the two spaces coincide under this map. The fundamental properties we distinguish when we say two topological spaces are isomorphic are the cardinality of the set and the topology it is endowed with.

Note that the distinguished property tied to the notion of isomorphism—the fundamental structure underlying each object in a category—in a specific setting (e.g., isomorphisms of sets) is fundamentally captured by specifying the morphisms in a category. In other words, somewhat unsurprisingly, the notions of morphism and isomorphism are inextricably linked. Once we specify the morphisms in a category we understand what an isomorphism is, and vice versa, though, given a notion of isomorphism, it is perhaps not always so immediately clear what the corresponding notion of morphism should be. In the analogy "isomorphism equals metaphor," morphisms are akin to weaker metaphors, or perhaps similes. A morphism between two objects in a category is a relation between them, but often a generic morphism tells us little about the way in which the objects relate. For example, given topological spaces X and Y, the constant morphisms taking the entirety of X to a single point of Y are continuous, but they tell us little about the way in which X and Y relate themselves. Likewise, given two objects, it is often the case that there exist trivial similes between them—there are manifold ways in which two things can be thought of as alike. Strengthening the comparison by way of metaphor allows us to go beyond knowing that the objects are just related: metaphor allows us to see the very way in which the objects are similar.

Categories themselves are mathematical objects, so a natural next question is whether there exists a "category of categories." All that is needed is a notion of a morphism between two categories, i.e., a "mapping" from one category to another. Equivalently, what are the fundamental properties that we distinguish in defining a category? Recall that there are two: composition of morphisms and the existence of an identity morphism from each object to itself. Thus, a morphism between two categories should respect both of these properties. Morphisms between categories are called *functors*.[1]

A (covariant) functor $\mathcal{F}$ between two categories C and $\mathcal{D}$ (usually denoted $\mathcal{F} : C \dashrightarrow \mathcal{D}$) is the data of:

(1) a function $\mathcal{F}$ from the objects of C to the objects of D;
(2) for each $X, Y \in C$, a function $\mathcal{F} : \mathrm{Mor}(X, Y) \to \mathrm{Mor}(\mathcal{F}(X), \mathcal{F}(Y))$ such that $\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$ and $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ for $f : X \to Y$ and $g : Y \to Z$.

We could have just have easily required that $\mathcal{F}$ take $\mathrm{Mor}(X, Y) \to \mathrm{Mor}(\mathcal{F}(Y), \mathcal{F}(X))$; we call such a functor *contravariant*. Note that a functor is exactly the desired notion of "morphism between categories." Moreover, every category comes naturally equipped with an identity functor taking each object and morphism to itself. If $\mathcal{F} : C \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to C$ are two functors such that $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathcal{D}}$ and $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_C$, then we say that the categories C and $\mathcal{D}$ are *equivalent*. In other words, C and $\mathcal{D}$ are isomorphic in the category of categories.

We illustrate the concept of functor via the aforementioned Yoneda Lemma. A category C is called *locally small* if, for any two objects X and Y of C, we have that $\mathrm{Mor}(X, Y)$ is a set (some authors include this condition in their definition of a category). Locally small categories are of interest because any locally small category C comes naturally equipped with a class of functors $C \dashrightarrow \mathrm{Set}$ indexed by the objects of C. Specifically, for any object X in C, there is a functor $\mathcal{F}_X : C \dashrightarrow \mathrm{Set}$ given by taking each object Y of C to the set $\mathrm{Mor}(X, Y)$. Yoneda's lemma, which is surprisingly easy to prove, states that the functor $\mathcal{F}_X : C \dashrightarrow \mathrm{Set}$ completely determines the object X, up to canonical isomorphism. Colloquially, the data of $\mathrm{Mor}(X, Y)$ for all objects Y in C completely determines X up to a natural isomorphism. Stating the lemma even more colloquially allows us to see that Yoneda's lemma is fundamentally Wittgensteinian: an object is entirely determined (up to natural isomorphism) by the way in which it relates to every other object in the category. Yoneda's lemma lends more credence to Leibniz's Law—that equal objects share all the same properties—and should at the very least further justify our natural inclination to understand and communicate this understanding of objects via metaphor.

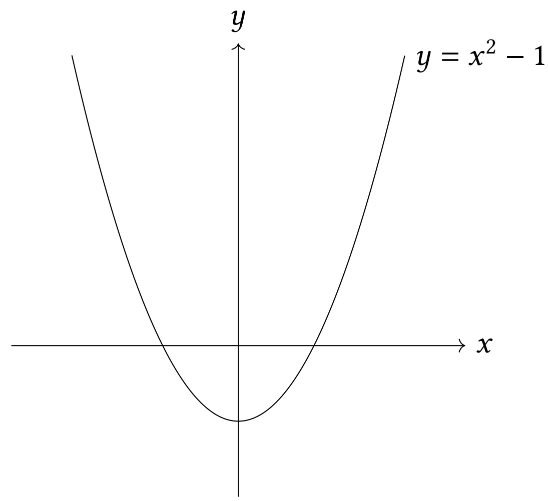
If we think of a category as the space in which a mathematical theory "lives," then functors between categories are analogies between two mathematical theories. Often, there are functors, or even equivalences, between seemingly unrelated categories. And just as a metaphor between two seemingly unrelated objects (e.g., between a cat and city smog) lets you view each of the objects in a new light, these unexpected categorical equivalences allow one to apply insights and intuition from one field to another, and ultimately help mathematicians discern what something truly *is*.

The Oxford Essential Mathematical Travel Dictionary: Algebra–Geometry and Geometry–Algebra

Translation is that which transforms everything so that nothing changes.
–Günter Grass

[1]Just as there are morphisms between categories, there is a notion of morphisms between functors—called *natural transformations*—that make the collection of functors from one category to another into a category itself! Continuing in this fashion, we arrive at a kind of meta-categorical structure called an ∞-category.

A metaphor between algebra and geometry that we learn in high school is the following: the algebraic equation $y = x^2 - 1$ can be thought of as the parabola given by plotting the points (x, y) satisfying $y = x^2 - 1$ in the plane.



As we will see in the sequel, this analogy is in fact an equivalence. The important (algebraic) properties of the equation imply all of the interesting (geometric) properties of the parabola, and vice versa. This link between algebra and geometry turns out to be an equivalence of categories—a metaphor between algebra and geometry. And what is perhaps so striking about this analogy is that algebra and geometry are seemingly irreconcilably different; the power of the comparison comes from the fact that our algebraic *intuitions* are so different from our geometric ones [8].

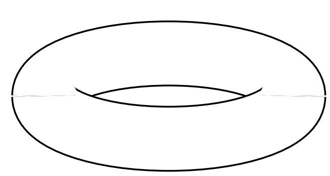
Consider the field of complex numbers, which we denote using $\mathbb{C}$. The set of ordered n-tuples of complex numbers is denoted $\mathbb{C}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{C}\}$; we may think of $\mathbb{C}^n$ as an n-dimensional space. Geometrically, we are interested in the simultaneous solutions $(a_1, \ldots, a_n) \in \mathbb{C}^n$ to a system of polynomials. In other words, if

$$f_1(x_1, \ldots, x_n), f_2(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)$$

are a system of m polynomials in the n variables $x_1, \ldots, x_n$ with complex coefficients, then we would like to consider the subset of points $(a_1, \ldots, a_n)$ of $\mathbb{C}^n$ such that

$$f_1(a_1, \ldots, a_n) = f_2(a_1, \ldots, a_n) = \cdots = f_m(a_1, \cdots, a_n) = 0.$$

We have already seen one example of such a subset—the parabola cut out by the equation $y-x^2+1$. Other famous examples include circles, ellipses, the other conic sections, as well as spheres, plane curves, tori (think doughnuts—pictured below), and quadric surfaces.



A subset V of $\mathbb{C}^n$ is called *algebraic* if it is the zero locus of some system of polynomial equations. These subsets of $\mathbb{C}^n$ are precisely the geometric objects we would like to consider. There are intuitive geometric properties that we can associate to each such subset, e.g., dimension, genus (number of holes), tangent space, irreducibility (think connectedness), etc. For example, the parabola $y = x^2 - 1$ is one-dimensional, since it locally looks like a line, whereas the torus pictured above locally looks two-dimensional. As we have seen, whenever we encounter a new family of mathematical objects, we should ask what it means to "map" between two such objects. In other words, how can we categorize algebraic subsets? A morphism between algebraic subsets $V \subset \mathbb{C}^n$ and $W \subset \mathbb{C}^m$ is a function $\varphi : V \longrightarrow W$ that is given by polynomials. In other words, there exist polynomials $f_1, \ldots, f_m$ in $\mathbb{C}[x_1, \ldots, x_n]$ such that

$$\varphi(p) = (f_1(p), \ldots, f_m(p))$$

for each point $p \in V$. This notion of morphism allows us to define the *category of complex algebraic subsets*, which we denote using C.

One easy example of a morphism in C is the following. Consider $\mathbb{C}$, and let P denote the vanishing locus of the polynomial $y - x^2 + 1$. We have seen this before: P is the parabola pictured previously—the points (x, y) in $\mathbb{C}^2$ satisfying $y = x^2 - 1$. There is an obvious morphism $\varphi : \mathbb{C} \longrightarrow \mathbb{C}^2$ given by $\varphi(t) = (t, t^2 - 1)$; simply projecting onto the x-axis gives us a morphism $\psi : V \longrightarrow \mathbb{C}$ given by $\psi(x, y) = x$ for a point (x, y) in V. Since $\varphi \circ \psi = \mathrm{id}_P$ and $\psi \circ \varphi = \mathrm{id}_{\mathbb{C}}$, it follows that P and C are isomorphic as algebraic subsets. This makes precise our geometric intuition that P "looks like" a line.

The label "algebraic" is not coincidental; in fact, it is quite suggestive. After all, the geometric objects that we are dealing with are cut out by algebraic, polynomial equations. Moreover, the appearance of the category C is suggestive of a functorial algebraic connection. So what is the corresponding "algebraic category" we will relate C to? And more fundamentally, why should we look to algebra in the first place? Why not deal with things purely geometrically?

Since it requires more substantive mathematical exposition, we defer answering the first question to the next paragraph. A partial answer to the latter question is the following: algebra is practical. Despite being motivated by geometry, it turns out that it can be very difficult to prove certain geometric properties of algebraic subsets purely using geometry. While our geometric intuition is oftentimes quite strong (e.g., the dimensions of the parabola are intuitively obvious), it turns out that the geometric definitions of relevant properties can be quite clunky and difficult to work with. On the other hand, while algebra may not be as intuitive, it is better-understood and easier to manipulate. Thus, in classical algebraic geometry, our goal is to take a geometric problem—understanding the properties of these shapes cut out by polynomials—and translate it into an algebra problem which we can solve. Although this connection between algebra and geometry was at first utilitarian, we will later see that there is something deeper and more fundamental lurking behind the scenes.

We now describe the aforementioned algebraic category which we will relate C to. Let $\mathbb{C}[x_1, \ldots, x_n]$ denote the set of polynomials in the n variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{C}$. For example, an element of $\mathbb{C}[x, y]$ might look like

$$f(x, y) = y^2 - x^3 + x - 1.$$

Polynomials are very similar to the integers $\mathbb{Z}$. Just as we can add and multiply integers, we can add and multiply polynomials; this endows $\mathbb{C}[x_1, \ldots, x_n]$ with *algebraic structure*. In particular, $\mathbb{C}[x_1, \ldots, x_n]$ is an example of a *ring*, which should be thought of as a set equipped with a way of adding and multiplying two elements together. In this way, rings are generalizations of the integers $\mathbb{Z}$. The addition and multiplication operations are required to satisfy the usual associative, commutative, and distributive properties, and in any ring there exist analogs of 0 and 1, i.e., additive and multiplicative identities, respectively. In a ring R, denote the identity elements by 0_R and 1_R. The collection of rings forms a category: given two rings R and S, a morphism of rings is simply a function $f : R \to S$ that preserves the ring addition, multiplication, and multiplicative identity. In other words, for all a and b in R, we must have that

$$f(a+b) = f(a) + f(b); \quad f(ab) = f(a)f(b); \quad f(1_R) = 1_S.$$

Let Ring denote the category of rings.

Before proceeding, we define an important algebraic substructure of a ring that will be crucial in connecting the algebra to the geometry. An *ideal* of a ring R is a subset I of R satisfying the following three properties:

(1) for all a and b in I, we have $a + b$ is also in I;
(2) a in I implies $-a$ is in I;
(3) for all a in I and r in R, we have $r \cdot a$ is in I.

In short, I is closed under addition, negation, and scaling by any element of R.

Now, how do we go about linking the geometry and algebra together? We have already seen that a set of polynomials gives rise to an algebraic subset, namely by considering their common zeros, but how do we go in the other direction? The answer comes to us via studying functions on our algebraic subset. First of all, intuitively, we see that the set of functions on an algebraic subset V—i.e., the morphisms $f : V \to \mathbb{C}$—form a ring. Two such functions f and g can be added together and multiplied by setting $(f+g)(p) = f(g)+g(p)$ and $(fg)(p) = f(p)g(p)$; the constant functions taking all of V to 0 and 1 are the additive and multiplicative identities, respectively. Given a polynomial $f(x_1, \ldots, x_n)$ in $\mathbb{C}[x_1, \ldots, x_n]$, we can treat f as a morphism $\mathbb{C}^n \to \mathbb{C}$: for each point $(a_1, \ldots, a_n)$ in $\mathbb{C}^n$, we can evaluate f at this point to get a complex number $f(a_1, \ldots, a_n)$. From this point of view, the polynomials $\mathbb{C}[x_1, \ldots, x_n]$ can be thought of as the set of functions on $\mathbb{C}^n$, which take as input a point in $\mathbb{C}^n$ and output a complex number. Importantly, two distinct polynomials give two distinct functions on $\mathbb{C}^n$. For an algebraic subset V of $\mathbb{C}^n$, we might likewise consider $\mathbb{C}[x_1, \ldots, x_n]$ to be the set of functions on V: for a polynomial f in $\mathbb{C}[x_1, \ldots, x_n]$ and point p in V, we can view f as a function on V by evaluating f at p, yielding a complex number $f(p)$. Yet there is something wrong with this point of view: it is not necessarily the case that two distinct polynomials will yield two distinct functions.

Consider the following concrete scenario. Let V be the parabola defined by $y = x^2 - 1$, as before, and consider the polynomials $\mathbb{C}[x, y]$. Let $f(x, y)$ be an arbitrary polynomial in $\mathbb{C}[x, y]$, and consider the point (a, b) in V. Since (a, b) lies on the parabola, we have that $b = a^2 - 1$ If we view f as a function on V, then we can evaluate f at (a, b) by plugging in $x = a$ and $y = b$, which yields the number $f(a, b)$. Now, consider the polynomial

$$f(x, y) + g(x, y) \cdot (y - x^2 + 1)$$

as a function on V, where we allow g to be any polynomial in $\mathbb{C}[x, y]$. Evaluating at (a, b), we see that

$$f(a, b) + g(a, b) \cdot (a - b^2 + 1) = f(a, b) + g(a, b) \cdot 0 = f(a, b),$$

where the first equality follows from the fact that $b = a^2 - 1$. Thus, despite being distinct polynomials in $\mathbb{C}[x, y]$, we see that $f(x, y)$ and $f(x, y) + g(x, y) \cdot (y - x^2 + 1)$ agree as *functions* on V. In fact, we see that any two polynomials f and g whose difference vanishes on V give identical functions on V. Now, suppose that V is a more general algebraic subset of $\mathbb{C}^n$. The set of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ which vanish on V forms an ideal of the ring $\mathbb{C}[x_1, \ldots, x_n]$ called the *vanishing ideal* of V. Indeed, we see that if f and g are polynomials such that $f(p) = 0$ and $g(p) = 0$ for all p in V, then $f + g$, $-f$, and $f \cdot h$, where h is an arbitrary polynomial, are all 0 on V. To sum up the problem at hand using this new terminology: two polynomials give the same function on V if and only if their difference lies in the vanishing ideal of V.

To fix this issue, we simply declare two polynomials f and g that agree on V to be equal. In other words, f and g are equal as functions on V if their difference lies in the vanishing ideal of V. From this point of view, we see that distinct equivalence classes of polynomials (i.e., polynomials whose difference is nonzero on V) give distinct functions on V. Thus, we see that the ring of functions on an algebraic subset V of $\mathbb{C}^n$ is simply the polynomials $\mathbb{C}[x_1, \ldots, x_n]$ up to this new notion of equivalence.[2] Denote this ring of functions on V by $\Gamma(V)$.

We have done a lot of work; we now see the payoff: a precise realization of the aforementioned metaphor between the functions describing a geometric shape and the shape itself. In the above, we have secretly been describing a contravariant functor Γ from the category C of algebraic subsets to Ring, which associates to each algebraic subset V its ring of functions $\Gamma(V)$. We have an association of objects, but what about morphisms? Given a morphism of algebraic subsets $\varphi : V \to W$, we get a corresponding morphism of rings, denoted $\varphi^* : \Gamma(W) \to \Gamma(V)$, taking a function f on W to the function $f \circ \varphi$ on V. Implicit in this association is the ideal of functions vanishing on V, since this set fundamentally determines what $\Gamma(V)$ *is*.[3]

It is natural to ask whether the functor Γ described above is an equivalence of categories. The short answer is not quite. The category Ring is, in a way, too big: given a generic ring, there is not a natural way to recover an algebraic subset of $\mathbb{C}^n$. However, by restricting the codomain Ring to be the subcategory $\mathcal{D}$ of finitely-generated, reduced $\mathbb{C}$-algebras, Γ becomes an equivalence of categories $C \dashrightarrow \mathcal{D}$.[4] Thus, this connection between algebra and geometry goes beyond just an intuitive correspondence—there is a deeper equivalence at play, which we will explore in the coming section.

It can be helpful to think of Γ as a sort of algebro-geometric dictionary, which allows us take concepts from geometry and translate them into algebra, and vice versa. The algebraic translations of geometric notions, such as dimension and irreducibility, are far easier to work with than their geometric counterparts. Likewise, given an algebraic problem, there is often helpful geometric intuition that can shed light on how to attempt a solution.[5] However, the fact that we can only translate back and forth between geometry and algebra is somewhat unsatisfying: this algebro-geometric isomorphism hints at the existence of some more intrinsic connection between the two subjects. In other words, we seek some object tor concept hat can simultaneously unify both the algebraic and geometric aspects of this theory. The idea in question is that of a *scheme*.

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[2] Without the language of quotients, this notion—that of the coordinate ring of an algebraic subset—gets quite messy. We are also going to ignore some technicalities entirely in this exposition: in particular, we are neglecting to discuss radical ideals or nilpotence.

[3] Those familiar with the notion of *quotient rings* will recognize that $\Gamma(V)$ is simply the quotient of $\mathbb{C}[x_1, \ldots, x_n]$ by the vanishing ideal.

[4] For the sake of accessibility and brevity, we will not further mention the words finitely-generated or reduced or the word algebra in this context.

[5] For a far better (not to mention more detailed) introduction to algebraic geometry, we refer to reader to two of the standard references: the textbooks by Hartshorne and Vakil [4, 9].

Picturing Primes

I can illustrate the second approach with the... image of a nut to be opened.
The first analogy that came to my mind is of immersing the nut in some softening liquid, and why not simply water? From time to time you rub so the liquid penetrates better, and otherwise you let time pass. The shell becomes more flexible through weeks and months—when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado!
–*Alexandre Grothendieck*, Récoltes et Semailles [3]

One of the simplest examples of an algebraic subset of $\mathbb{C}^n$ is a set containing a singular point of $\mathbb{C}^n$. The corresponding vanishing ideal of $\mathbb{C}[x_1, \ldots, x_n]$ turns out to be *maximal*, in that the only ideal containing it is the entire ring, $\mathbb{C}[x_1, \ldots, x_n]$. Conversely, the zero set of a maximal ideal of $\mathbb{C}[x_1, \ldots, x_n]$ is a single point in $\mathbb{C}^n$. Hence, we see that the points making up the geometric space $\mathbb{C}^n$ itself can be viewed purely algebraically as a ideals. Herein lies the deeper idea underlying the algebro-geometric metaphor we established in the previous section. In order to fully distill the algebra and geometry into a singular idea, we must bake together the algebra and geometry into an object that can simultaneously "see" both at once.

In the previous section, we saw that in order to understand the geometry of an algebraic subset of $\mathbb{C}^n$ it is equivalent to understand its associated algebraic ring of functions on the space. Unfortunately, not every ring of functions corresponds to an algebraic subset of $\mathbb{C}^n$. To fix this, given a ring R, we simply construct a space whose associated ring of function is R itself. We take the points of this space to be the set of *prime ideals* of R, which contains the maximal ideals of R.[6] This set, denoted $\text{Spec}(R)$, can be endowed with a natural topology. Moreover, there is also a way in which we see that R is the ring of functions on $\text{Spec}(R)$. By gluing together several of these spaces $\text{Spec}(R)$, we get a *scheme*.

Considering $\text{Spec}(\mathbb{C}[x_1, \ldots, x_n])$, we see that the resulting space is effectively an enrichment of $\mathbb{C}^n$, which tells us not only about $\mathbb{C}^n$ itself but, at the same time, its algebraic subsets. Though extremely difficult to understand at first, this simultaneous view of both the algebra and geometry condenses the data of the isomorphism of categories from the previous section into one object, distilling our algebraic and geometric perspectives and intuitions into a singularity: the scheme.

The Rising Sea

A different image came to me a few weeks ago.
The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration… the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it... yet it finally surrounds the resistant substance.
–*Alexandre Grothendieck*, Récoltes et Semailles [3]

In the above, Grothendieck suggests a metaphor in answer to the question posed in the introduction: how does one go about creating definitions in mathematics? Grothendieck likens the

[6]Prime ideals are a direct generalization of integral primes. In certain rings, prime ideals behave exactly like prime numbers, and any ideal can be factored uniquely in to a product of prime ideals. Geometrically, the prime ideals of $\mathbb{C}[x_1, \ldots, x_n]$ correspond bijectively to *irreducible* algebraic subsets of $\mathbb{C}^n$, which are algebraic subsets which cannot be broken down into a union of smaller algebraic subsets. With this in mind, we see that the construction of $\text{Spec}(R)$ unifies the algebraic, geometric, and number-theoretic intuitions via the motto: "primes are points."

problem of understanding what a mathematical concept fundamentally *is* to the problem of surrounding a monolithic mound of earth with water. The water around the rock represents one's understanding of the rock. At first, one's understanding is dwarfed by the vastness of the problem, and "the water is so far off you hardly hear it." Yet as one studies further, the sea rises, and the depth of one's understanding grows. Eventually, the ocean surrounds and encompasses the once-daunting mound of earth.

The art of creating a metaphor or isomorphism follows this same process. One starts off with a simple, seemingly trivial connection, such as the link between an algebraic equation and its graph. At first, one might ignore the connection, or perhaps think one understands it fully. But, over time, one begins to realize that there is a deeper connection, and, with practice—by immersing oneself in the concept—one is able to penetrate the layers of resistance and navigate with ease. Once idea and understanding merge, what was once impenetrable is knowable.

Isomorphism and metaphor are like waves trying to reach the seemingly unknowable, with each connection and comparison trying to get to the essence of an idea or object. Whereas analogy is more pedestrian in its observation of some shared property, metaphor and isomorphism invite creativity and artistry in how the connection is made. This creative process involves choosing which shared property to distinguish, which by necessity gets to the heart of the relationship between the two objects or ideas. And if we adopt Wittgenstein's view that "objects are determined by the way in which they relate to every other object," then we see that understanding the ways in which one object relates to another is equivalent to pursuing a definitional understanding of what something truly is.

Still, metaphor and isomorphism are not merely means to an end. They are more than tools to explore how ideas and objects are related or are equal. The artistry in the comparisons can be beautiful and expressive in their own right. Because the poet and mathematician must creatively choose which similar property to distinguish and then determine how to express or prove the equality of that property, the very nature of metaphor and isomorphism allows us to get to the essence of the objects compared and the notion of equality itself. Both metaphor and isomorphism show us that equality as a property is not about things being identical; rather the notion of equality as realized in metaphor and isomorphism is about the range of possibility of relatedness between ideas or objects. Thus, while it may seem that connecting two objects or ideas to each other fixes them in a limited way—by focusing on how they are related and disregarding ways in which they are different—in reality, it allows them to be viewed more expansively with an eye to appreciating their quintessence and beauty.

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